



The Multivariable I-Function and Large Deflection of a Clamped Circular Plate

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1. Introduction

In the present analysis, we have discussed the large deflection, plane and radial displacement and the bending stresses for a clamped circular plate under axially symmetric non-uniform load by se of Berger's approximate method. In the end, some particular cases and the family of load shapes have been discussed.

Notations used

$$D = \frac{Eh^3}{12(1-\theta^2)}, \text{ the flexural rigidity of the plate,}$$

h = thickness of the plate

E = Young's modulus,

θ = Poisson's ratio

= Lateral Displacement

= small deflection

u = radial displacement,

σ_r and σ_ϕ = axially symmetric non-uniform load.

Statement of the problem and governing equations

Let us consider a circular plate of radius a , thickness h and flexural rigidity D whose edge is clamped, We also assume that an axially symmetric non-uniform load $f(r)$, given by

$$f(r) = K_0 \left[1 - (r/a)^k \right]^{\mu} I_{\substack{o, o : \dots, o, o(m', n'); \dots; (m^{(N)}, n^{(N)}) \\ P_2, q_2 : \dots, P_r, q_r : [p', q']; \dots [p^{(N)}, q^{(N)}]}} \left[z_1 \left\{ (r/a)^k - 1 \right\}^{\delta_1} : \dots, \left\{ z_n (r/a)^k - 1 \right\}^{\delta_n} \right] \quad (1.1)$$

Where K_0 is constant and the I-function in (1.1) is the multivariable I- function of N variables defined by Prasad [1], is imposed to the plate. If we neglect the strain energy due to second variant in the middle plane of the plate, then following the Berger's approximate method [2] the equation for large deflection due to an externally applied load f , for circular plate, can be given as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - c^2 w \right) = \frac{f(r)}{D} = F(r) \quad \text{(say)} \quad (1.2)$$

Where c is the normalized constant of integration given by



$$\frac{du}{dr} + \frac{u}{r} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 = \frac{c^2 h^2}{12}, \quad (1.3)$$

Where w, u, h and D are specified above. Since the circumference of the plate is clamped, the boundary conditions, therefore, for the problems are

$$w = 0 = \frac{dw}{dr} \text{ where } r = a$$

and $u = 0$ where $r = a$ (1.4)

2. **Solution of the problem**

Let us suppose the solution of the problem in the form

$$W = \sum_j A_j \left[J_0(r\xi_j) - J_0(a\xi_j) \right], \quad (2.1)$$

ξ_j being the jth root of equation

$$J_1(a\xi_j) = 0 \quad (2.2)$$

And $J_\nu(x)$ is Bessel function

Obviously, the boundary condition (1.4) are satisfied by the solution assumed in (2.1). Combining equations (2.1) and (1.2) with the use of differential formula and recurrence relation for Bessel functions (3 Vol II Ch VII), we arrive at

$$\sum_j A_j \xi_j^2 (c^2 + \xi_j^2) J_0(r\xi_j) = F(r) \quad (2.3)$$

Where A_j 's are the constant to be determined. Multiply both sides of (2.3) by $r J_0(r\xi_j)$ and integrate with respect to r between limits 0 to a and utilize the orthogonal property of Bessel function [4, Vol. II, p. 70] to have

$$A_j = \frac{2}{a^2 \xi_j^2 (c^2 + \xi_j^2) [J_0(a\xi_j)]^2} \int_0^a r J_0(r\xi_j) F(r) dr. \quad (2.4)$$

A general solution to the problem may, therefore, be given in the form

$$W = \frac{2}{a^2} \sum_j \frac{[J_0(r\xi_j) - J_0(a\xi_j)]}{\xi_j^2 (c^2 + \xi_j^2) [J_0(a\xi_j)]^2} \int_0^a r J_0(r\xi_j) F(r) dr. \quad (2.5)$$

Provided that $F(r)$ is such that the integral and the series in (2.5) are convergent. Further, appeal to the result.

$$\int_0^1 y^\rho (1-y)^\mu I_{\rho, \rho: \dots, \rho: (m', n'); \dots, (m^{(N)}, n^{(N)})} \left[z_1 (1-y) \delta^1 \right]$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l (x/2)^{\nu+2l} \Gamma\left(\rho + \frac{\nu+2l}{k} + 1\right)}{\Gamma(\nu+l+1)} I_{\rho_2, q_2: \dots, \rho_r, q_r: [p', q']; \dots, p^{(N)}, q^{(N)}} \left[z_1 (1-y) \delta^1 \right]$$



$(m^{(N)}, n^{(N)})$

$$p_r + 1, q_r + 1 : [p', q'] ; \dots ; [p^{(N)}, q^{(N)}] \left[\begin{matrix} (a_{2j}; \alpha_{2j}'', \alpha_{2j}''')_{1, p_2} : \dots : \\ (b_{2j}; \beta_{2j}', \beta_{2j}''')_{1, q_2} : \dots : \end{matrix} \right.$$

$$\left(a_{r-1j}; \alpha'_{r-1j} : \dots, \alpha_{r-1j}^{(r-1)} \right)_{1, p_{r-1}} : (1 - \mu; \delta_1 : \dots, \delta_N),$$

$$b_{r-1j}; \beta'_{r-1j}, \dots, \beta_{r-1j}^{(r-1)}_{1, q_{r-1}} : (b_{Nj}; \beta'_{Nj}, \dots, \beta_{Nj}^{(N)})_{1, q_N},$$

$$(a_{Nj}; \alpha'_{Nj} : \dots, \alpha_{Nj}^{(N)})_{1, p_N} : (a_j, \alpha_j)_{1, p'} ; \dots ; (a_j^{(N)}, \alpha_j^{(N)})_{1, p^{(N)}}$$

$$\left(-\rho - \frac{\nu + 2\rho}{k} - \mu - 1; \delta_1, \dots, \delta_N \right); (b'_j, \beta'_j)_{1, q'} ; \dots ; (b_j^{(N)}, \beta_j^{(N)})_{1, q^{(N)}} \\ \left[Z_1, \dots, Z_n \right]$$

(2.6)

where

$$\operatorname{Re}(\rho + \nu/k) > -1, \operatorname{Re} \left(\mu + 1 + \sum_{i=1}^N \delta_i \alpha_i \right) > 0, |\arg z_i| < \frac{1}{2} \pi \nu_i$$

($i=1, \dots, N$), α_i and U_i are given in [1], for $i = 1, \dots, N$ leads to

$$W = \frac{2k_o}{KD} \sum_j \frac{[J_o(r\xi_j) - J_o(a\xi_j)]}{\xi_j^2 (c^2 + \xi_j^2) [J_o(a\xi_j)]^2} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l \left(\frac{1}{2} a \xi_j \right)^{2l}}{l! \Gamma(l+1)} \right\}$$

$$\Gamma \left(\frac{2^l + 2}{K} \right) I_{\rho, o : \dots, o, o : o, 1 : (m', n')} ; \dots ; \left(m^{(N)}, n^{(N)} \right) \\ p_2, q_2 : \dots : p_{N-1}, q_{N-1} : p_N + 1, q_N + 1 : [p', q'] ; \dots ; [p^{(N)}, q^{(N)}]$$

$$\left[\begin{matrix} \left(a_{2j}; \alpha'_{2j}, \alpha''_{2j} \right)_{1, p_2} : \dots : \left(a_{N-1j}; \alpha'_{N-1j}, \dots, \alpha_{N-1j}^{(N-1)} \right)_{1, p_{N-1}} : \\ \left(b_{2j}; \beta'_{2j}, \beta''_{2j} \right)_{1, q_2} : \dots : \left(b_{N-1j}; \beta'_{N-1j}, \dots, \beta_{N-1j}^{(N-1)} \right)_{1, q_{N-1}} : \end{matrix} \right.$$

$$\left(-\mu; \delta_1, \dots, \delta_N \right), (a_{Nj}; \alpha'_{Nj}, \dots, \alpha_{Nj}^{(N)})_{1, p_N} : (a_j, \alpha_j)_{1, p'} ; \dots ;$$

$$\left(b_{Nj}; \beta'_{Nj}, \dots, \beta_{Nj}^{(N)} \right)_{1, q_N} \left(-\frac{2}{k} - \frac{2l}{k} - \mu; \delta_1 : \dots, \delta_N \right) : (b'_j, \beta'_j)_{1, q'} ; \dots ;$$



$$\left. \begin{matrix} (a_j^{(N)}, \alpha_j^{(N)})_{1,p^{(N)}} \\ (b_j^{(N)}, \beta_j^{(N)})_{1,q^{(N)}} \end{matrix} \right\{ z_1, \dots, z_n \} \quad (2.7)$$

provided that the condition given in (2.6) $\left(\text{for } P = \frac{2}{k} - 1, u = 0 \right)$ are satisfied.

To obtain the expression for radial displacement u , introducing (2.1) into (1.3) and integrating with respect to r in view of result [4, vol. II, Chap. VII, p. 90(9), (10)], we arrive at

$$\begin{aligned} ru &= \frac{c^2 h^2 r^2}{24} - \frac{1}{2} \sum_{j=1}^{\infty} A_j^2 \xi_j^2 \frac{r^2}{2} \left[\{J_1(r\xi_j)\}^2 - J_0(r\xi_j)J_2(r\xi_j) \right] \\ &- \frac{1}{2} \sum_{t=0}^{\infty} \sum_{\substack{j=1 \\ t \neq j}}^{\infty} A_j A_t \xi_j \xi_t r \left(\xi_j^2 - \xi_t^2 \right)^{-1} \left[\xi_j J_2(r\xi_j) \right. \\ &\left. J_1(r\xi_t) - \xi_t J_2(r\xi_t) - J_1(r\xi_j) \right] + C_1, \end{aligned} \quad (2.8)$$

Where c_1 is constant of integration. The boundary condition (1.4), consequently demands

$$C_1 = \frac{c^2 h^2 a^2}{24} + \frac{1}{4} \sum_{j=1}^{\infty} A_j^2 \xi_j^2 a^2 J_2(a\xi_j)^2 \quad (2.9)$$

If the deflection in the plate is small, then letting $c = 0$, the differential equation (1.2) corresponds to that for the small deflection cases and the solution (2.7) then leads to

$$W = \frac{2K_0}{KD} \sum_j \frac{[J_0(r\xi_j) - J_0(a\xi_j)]}{\xi_j^4 [J_0(a\xi_j)]^2} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l (a\xi_j/2)^{2l}}{l! \Gamma(l+1)} \right.$$

$$\left. \Gamma \frac{(2^l + 2)}{k} I_{o, o : \dots : o, o : o, 1 : (m', n') ; \dots ; (m^{(r)}, n^{(r)})} \right. \\ \left. p_2, q_2 : \dots, p_{r-1}, q_{r-1} : p_r + 1, q_r + 1 : [p', q'] ; \dots ; [p^{(r)}, q^{(r)}] \right]$$

$$\left[\begin{matrix} \left(a_{2j} ; \alpha'_{2j}, \alpha''_{2j} \right)_{1,p_2} : \dots : \left(a_{N-1j} ; \alpha'_{N-1j}, \dots, \alpha_{N-1j}^{(N-1)} \right)_{1,p_{N-1}} : \\ \left(b_{2j} ; \beta'_{2j}, \beta''_{2j} \right)_{1,q_2} : \dots : \left(b_{N-1j} ; \beta'_{N-1j}, \dots, \beta_{N-1j}^{(N-1)} \right)_{1,q_{N-1}} : \end{matrix} \right]$$



And at the edge ($r=a$) to be

$$(\sigma_r) = \frac{12k_0}{kh^2} \sum_j \frac{1}{(c^2 + \xi_j^2) J_0(a\xi_j)} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l \left(\frac{a}{2} \xi_j\right)^{2l}}{l! \Gamma(l+1)} \Gamma\left(\frac{2l+2}{k}\right) \right.$$

$$\left. I_{o,o,\dots,o,o,1:(m',n');\dots;(m^{(N)},n^{(N)})} \right.$$

$$\left. p_2, q_2, \dots, p_{N-1}, q_{N-1} : p_N + 1, q_N + 1 : [p', q']; \dots; [p^{(N)}, q^{(N)}] \right.$$

$$\left. \left[\begin{aligned}
 &(a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2} : \dots : (a_{N-1j}; \alpha'_{N-1j}, \dots, \alpha_{N-1j}^{(N-1)})_{1,p_{N-1}} : (-\mu; \delta_1, \dots, \delta_N) \\
 &(b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2} : \dots : (b_{N-1j}; \beta'_{N-1j}, \dots, \beta_{N-1j}^{(N-1)})_{1,q_{N-1}} : (b_{Nj}; \beta'_{Nj}, \dots, \beta_{Nj}^{(N)}) \\
 &(a_{Nj}; \alpha'_{Nj}, \dots, \alpha_{Nj}^{(N)})_{1,p_N} : (a'_j, \alpha'_j)_{1,p'}; \dots; (a_j^{(N)}, \alpha_j^{(N)})_{1,p} \\
 &(-\frac{2}{k} - \frac{2l}{k} - \mu; \delta_1, \dots, \delta_N) : (b'_j, \beta'_j)_{1,q'}; \dots; (b_j^{(N)}, \beta_j^{(N)})_{1,q^{(N)}} \Big| Z_1, \dots, Z_n \Big] \right\}$$

$$\tag{2.14}$$

$$(\sigma_\phi)_{r=a} = \theta(\sigma_r)_{r=a} \tag{2.15}$$

3. Particular cases and conclusions

(i) W_0 , the large deflection at the centre may be deduced by putting $r = 0$ in (2.7) and is found to be

$$W_0 = \frac{2k_0}{kD} \sum_j \frac{[1 - J_0(a\xi_j)]}{\xi_j^2 (c^2 + \xi_j^2) [J_0(a\xi_j)]^p} B(\xi_j) \tag{3.1}$$

Where $B(\xi_j)$, for space brevity, stands for the infinite series of the multivariable H-function falling in (2.7).

(ii) whereas the small deflection at the centre of plate is given by

$$W_0 = \frac{2k_0}{kD} \sum_j \frac{[1 - J_0(a\xi_j)]}{\xi_j^4 [J_0(a\xi_j)]^p} B(\xi_j). \tag{3.2}$$



(iii) If the normal pressure $f(r)$ is of the form

$$f(r) = K_0 \left[1 - (r/a)^k \right]^\mu \exp \left\{ -z \left((r/a)^k - 1 \right)^\delta \right\} \quad (3.3)$$

Then, in the view of connecting formula (5, p. 151), the deflection of the plate is given by

$$W = \frac{2k_0}{kD} \sum_j \frac{[J_0(r\xi_j) - J_0(a\xi_j)]}{\xi_j [c^2 + \xi_j^2] [J_0(a\xi_j)]^2} \sum_{l=0}^{\infty} \frac{(-1)^l (a\xi_j/2)^2}{l! \Gamma(l+1)} \Gamma\left(\frac{2^l+2}{k}\right) H_{1,2}^{1,1} \left[Z \left| \begin{matrix} (-\mu, \delta) \\ (0,1), \left(-\frac{2}{k} - \frac{2^l}{k} - \mu, \delta\right) \end{matrix} \right. \right], \quad (3.4)$$

Where $H_{p,q}^{m,n}[x]$ is the Fox's H-Function, provided that the corresponding conditions are satisfied.

(iv) If the load shape is assumed of the form

$$f(r) = K_0 \left[1 - (r/a)^k \right]^\mu \sin \left\{ 2z^{\frac{1}{2}} \left((r/a)^k - 1 \right)^{\delta/2} \right\}, \quad (3.5)$$

The expression for deflection, in this case may be shown to be

$$W = \frac{2k_0}{kD} \pi^{\frac{1}{2}} \sum_j \frac{[J_0(r\xi_j) - J_0(a\xi_j)]}{\xi_j^2 (c^2 + \xi_j^2) [J_0(a\xi_j)]^2} \sum_{l=0}^{\infty} \frac{(-1)^l (a\xi_j/2)^2}{l! \Gamma(l+1)} \Gamma\left(\frac{2^l+2}{K}\right) H_{1,3}^{1,1} \left[Z \left| \begin{matrix} (-\mu, \delta) \\ \left(\frac{1}{2}, 1\right), (0,2), \left(-\frac{2}{k} - \frac{2^l}{k} - \mu - \delta\right) \end{matrix} \right. \right], \quad (3.6)$$

Provided that the right side of (3.6) exists.

4. Analysis of the family of load shapes

If $p_2 = p_3 = \dots = p_r = 0 = q_2 = q_3 = \dots = q_r$, $N=1$

(i.e. H- function is of single variable), $m' = 1 = z_1 = \delta$, $1 =$

$\alpha'_1 = \alpha'_2 = \beta'_1 = \beta'_2, n' = 2 = p' = q', K_0 = K'_0 / \Gamma[b - \nu] a'_1 = 1 + \nu,$

$a'_2 = -\nu - \alpha - \beta, b'_1 = o, b'_2 = \alpha,$ then written in the limit when $b \rightarrow 0$, we have from (1.1)



$$f(r) = K_0' [1 - (r/a)^k]^\mu \frac{(v + \alpha + \beta + 1)}{\Gamma(1 + \alpha)} {}_2F_1[-v, v + \alpha + \beta + 1; +1; 1 - (r/a)^k] \quad (4.1)$$

Now, we shall discuss the following cases:

(i) If $v = 1; \alpha, \beta, \mu > 0$, then

$$f(r) = \frac{K_0' \Gamma[\alpha + \beta + 2]}{\Gamma(1 + \alpha)} [1 - (r/a)^k]^\mu \left[1 - \frac{\alpha + \beta + 2}{1 + \alpha} (1 - (r/a)^k) \right] \quad (4.2)$$

Equation (4.2) indicates that $f(r) = 0$ when $r = a$ and

$$(or) \quad \gamma = a \left(\frac{\beta + 1}{\alpha + \beta + 2} \right)^{1/k} \quad (4.3)$$

Evidently, for positive k , $f(r)$ is negative or positive according as

$$0 \leq r \leq a \left(\frac{\beta + 1}{\alpha + \beta + 2} \right)^{1/k} \quad or \quad a \left(\frac{\beta + 1}{\alpha + \beta + 2} \right)^{1/k} < r < a.$$

We, therefore, conclude that normal pressure over the plate are bounded by the circle

$0 \leq r \leq a \left(\frac{\beta + 1}{\alpha + \beta + 2} \right)^{1/k}$ is acting in -ve direction of z-axis (i.e. onward) whereas in the annular region $a \left(\frac{\beta + 1}{\alpha + \beta + 2} \right)^{1/k} < r < a$, it is acting in the +ve direction of z-axis (downward).

(ii) If we take $v = \alpha = 0 = \beta, \mu > 0$, then

$$f(r) = K_0' [1 - (r/a)^k]^\mu. \quad (4.4)$$

Equation (4.4), for positive K ($K \neq 0$), represents an axially symmetric pressure distribution acting in the positive direction (if $K_0' > 0$) of different intensities for different values of parameters K and μ .

(iii) Finally, if $v = 0 = \alpha = \beta = \mu$, then

$$F(r) = K_0' \quad (4.5)$$

Which stands for a uniform force of magnitude K_0' over the plate.

Remark: Evidently, the problem considered here is not without practical interest as we may have various non-uniform distributions, of thrust at pts of contact when heavy bodies of different shapes are put on the plate. Also in many practical problems, when the normal force is given by elementary functions, we can find the solution from the mains result (2.7) and plot various graphs.



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