The Miltivariable I-Function with Application to Temperature Distribution in Non-Homogeneous Moving Bar

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1.0. Introduction

In the present paper, we have discussed the temperature distribution in non-homogeneous moving bar. We use some results involving meijer's G-function (1) and the multivariable i- function defined by Prasad (2) to obtain the most general solution.

1. Statement of the problem and governing equations

To exhibit the application of maljer's G-function and the multivariable i-function in the problems of physical sciences, we consider a non-homogeneous bar moving in direction of its length (X-axis) between the limits x= -1 to x= +1. The bar is supposed to be so thin that the temperature at all points of the section may be taken to be the same. We also assume the conductivity and the velocity of the bar to be variable and that these are proportional to $(1=x^2)$ and $[(\propto -\beta+\gamma+(\alpha+\beta-\gamma)x)]$ where α,β,γ are constants. Lateral surface of the bar and its two ends are supposed to be insulated. The problem is thus one of linear flow in which the temperature is specified by the time and the distance, x, measured along the bar.

The differential equation satisfied by the temperature $\theta(x,t)$ at any time t in a uniform bar with conductivity K, density specific heat c and moving with velocity u in the direction of its length, is given by carslaw and jaeger $[3, P.127]$ as

1.1
$$\frac{k}{\rho c}\frac{\partial^2\theta}{\partial x^2}-u\frac{\partial\theta}{\partial x}-\frac{\partial\theta}{\partial t}=0$$

Now instead of constant conductivity and constant velocity of the bar with variable conductivity $k_0(1-x^2)$ and variable velocity $K[\alpha-\beta+\gamma+(\alpha-\gamma+\beta)x]$, where $K=k_0/\rho e$,

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > r-1$; K_0 ,α, β being all constants, the differential equation (1.1) reduces to

$$\frac{I}{K}\frac{\partial\theta}{\partial t}=(1-x^{2})\frac{\partial^{2}\theta}{\partial x^{2}}-[\beta-\lambda-\gamma-(\lambda-\gamma+\beta+2)]\frac{\partial\theta}{\partial x}$$

To obtain the most general solution, we assume the initial condition for the problem in the form

$$\theta(x,0) = f(x) = (1-x)^{\rho}(1+x)^{\sigma} \quad I^{0,n_2:\ldots:0,n_r:}_{p_2,q_2:\ldots:p_r,q_r}$$

$$\begin{matrix}(m',n');\ldots;(m^{(r)},n^{(r)})\\(p',q'):\ldots;(p^{(r)},q^{(r)})\end{matrix}\left[Z_1(1-x)^{\delta_1}(1+x)^{\mu_1},\ldots,Z_r(1-x)^{\delta r}(1+x^2)\right] \quad \ldots.\ 1.3$$

Where the I-function in (1.3) is multivariable I-function defined by Prasad (2). The following result will be used in the sequel:

(i)

$$\int_{-1}^{1}(1-x)^{P}(1+x)^{\sigma}G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\upsilon,-\upsilon-\alpha-\beta\\0,-\alpha\end{matrix}\right]dx$$

$$=2^{\rho+\sigma+1}\Gamma(\sigma+1)\sum_{N=0}^{\infty}\frac{\Gamma(-\upsilon+N)\Gamma(\upsilon+\alpha+\beta+1+N)\Gamma(\rho+N+1)}{N!\Gamma(N+\alpha+1)\,\Gamma(\rho+\sigma+2+N)} \quad (1.4)$$

Provided that Re (ρ) > -1, Re (σ) > -1.

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\upsilon,-\upsilon-\alpha-\beta\\0,-\alpha\end{matrix}\right]$$

(ii) $$G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\mu,-\mu-\alpha-\beta\\0,\alpha\end{matrix}\right]dx=0$$

when $\mu\neq\upsilon$, $\mathrm{Re}(\alpha)>-1;\mathrm{Re}(\beta)>-1.$

(1.5)

$$\int_{-1}^{1}(1-x)^{P}(1+x)^{\beta}G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\upsilon,-\upsilon-\alpha-\beta\\0,-\alpha\end{matrix}\right]d$$

(iii) $$G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\upsilon,-\upsilon-\rho-\beta\\0,\rho\end{matrix}\right]dx$$

$$=\frac{2^{\rho+\beta+1}\Gamma(\beta+\upsilon+1)\Gamma(\alpha+\beta+2\upsilon+1)\Gamma(\rho+\beta+\upsilon+1)}{\upsilon!\Gamma(\rho+\beta+2\upsilon+2)}\frac{[\Gamma(-\upsilon)\Gamma(\upsilon+1)]^{2}}{\Gamma(\upsilon+\alpha+1)}$$

Provided that Re (ρ)> -1; Re (β) >-1. (1.6)

(iv) $$\int_{-1}^{1}(1-x)^{P}(1+x)^{\sigma}G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\Big|\begin{matrix}1+\upsilon,-\upsilon-\alpha-\beta\\0,-\alpha\end{matrix}\right]$$

$$I^{0,n_2:\ldots:0,n_r:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p_2,q_2:\ldots:p_r,q_r:(p',q');\ldots;[p^{(r)},q^{(r)}]}\Big[Z_1(1-x)^{\delta_1}$$

$$(1+x)^{\mu_1},\ldots,Z_r(1-x)^{\delta r}(1+x)^{\mu_r}\Big]\,dx$$

$$=2^{\rho+\sigma+1}\sum_{n=0}^{\infty}\frac{\Gamma(\upsilon+\alpha+\beta+1)\Gamma(-\nu+N)}{N!\Gamma(N+\alpha+1)}I^{0,n_2:\ldots o,n_{r-1}:o,n_r+2:}_{p_2,q_2:\ldots;p_{r-1}q_{r-1}:p_r+2,q_r+1:}$$

$$(m',n');\ldots;(m^{(r)},n^{(r)})\left[\begin{matrix}(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1},p_2:\ldots:\\ \\ b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}\overset{\ldots\ldots}{:}\end{matrix}\right]$$

$$[p',q'];\ldots;(p^{(r)},q^{(r)}]$$

$$(a_{\overline{r-1j}};\alpha^{1}_{\overline{r-1j}},\ldots,\alpha^{(r-1)}_{\overline{(r-1)j}}):(-\sigma;\mu_1,\ldots,\mu_r),(-\rho-N;\delta_1,\ldots,)$$

$$b_{\overline{r-1j}};\beta'_{\overline{r-1j}},\ldots,\beta^{(r-1)}_{\overline{r-1j}})_{1,q_{r-1}}:(b_{rj}:\beta'_{rj},\ldots,\beta_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}$$

$$(a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,pr}:(a'_{j},\alpha'_{j})_{1,p'};\ldots;$$

$$(-\sigma-\rho-1-N;\delta_1+\mu_1,\ldots,\delta_r+\mu_r):(b'_j,\beta'_j)_{1,q};\ldots,$$

$$(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\Big|Z_1\,2^{\delta_1+\mu_1},\ldots,Z_r\,2^{\delta r+\mu r}\Big],$$

$$(b_j^{(r)},\beta_j^{(r)})_{1,p^{(r)}}$$

Provided that Re $(\rho+\sum_{i=1}^{r}\alpha_1\mu_1)>-1,$

$\mathrm{Re}(\sigma+\sum_{i=1}^{r}\alpha_i\mu_i)>-1;\delta_1\geq 0\,(i=1,\ldots,r),$

$\mu_r>o\,(\text{or }\mu_i>o,\delta_1>o),|\arg Z_i|<\frac{1}{2}\pi U_i$

$(U_i>0)$ for $i=1,\ldots,r$ where α_i, U_i are given by Prasad (2).

2. Solution and analysis

Let us assume the solution of the differential equation (1.2) in the form

$\theta = X(x)\ T(t),$

The differential equation (1.2), then reduces to

$$\frac{1}{KT}\frac{dT}{dt} = \frac{1}{x}\left[(1-x^2)\frac{d^2x}{dx^2} + \{\beta - \alpha - \gamma - (\alpha - \gamma + \beta + 2)\times\}\frac{dX}{dx}\right]$$

(2.2)

X and T being explicit functions, each side of (2.2) equals a constant. Assuming the constant to be $-\upsilon(\upsilon + \alpha - \gamma - (\beta - \alpha - \gamma + \beta + 1)$,

Clearly we have

$$(1-x^2)\frac{d^2X}{dx^2} + \{\beta - \alpha - \gamma - (\alpha - \gamma + \beta + 2)x\}$$

$$\frac{dX}{dx} + \upsilon(\upsilon + \alpha - \gamma + \beta + 1)X = 0 \tag{2.3}$$

and

$$\frac{dT}{dt} + K\upsilon(\upsilon + \alpha - \gamma + \beta + 1)T = 0 \tag{2.4}$$

It is easy to verify [4, p. 13(1.5.1)] that the solution of equation (2.3) is

$$X = G_1 G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma - \upsilon - \alpha - \beta \\ 0, -\alpha\end{matrix}\right]$$

(2.5)

Where C_1 is constant. Also, a solution of the differential equation(2.4)is

$$T = C_2 \exp\{-K_v(\upsilon + \alpha - \gamma + \beta + 1)t\} \tag{2.6}$$

C_2 being constant of integration. A general solution, therefore, of the equation (1.2) is

$$\theta(x,t) = \sum_{\upsilon=0}^{\infty} A_v \exp\{-K_\upsilon(\upsilon + \alpha - \gamma + \beta + 1)\} \tag{2.7}$$

$$G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma - \upsilon - \alpha - \beta \\ 0, \alpha\end{matrix}\right]$$

Where A υ's are constants. To determine these constants, applying the initial condition (1.) in (2.7), we arrive at

$$f(x) = \sum_{\upsilon=0}^{\infty} A_\upsilon \; G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma-\upsilon-\alpha-\beta \\ 0, \alpha\end{matrix}\right]$$

(2.8)

Now we multiply both sides of (2.8) by

$$(1-x)^{\alpha}(1+x)^{\beta-\gamma} \; G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma-\upsilon-\alpha-\beta \\ 0, \alpha\end{matrix}\right]$$

And integrate, thereafter, with respect to x between the limits x=-1 and x = 1. In the right side we interchange the order of integration and summation which is justifiable due to absolute convergence of the integral and the series involved therein. And finally appeal to the result (1.5) and (1.6) consequently, leads us to

$$A_\upsilon = \frac{2^{\gamma-\alpha-\beta-1}\Gamma(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\upsilon+\alpha+1)}{\Gamma(\beta+\upsilon+1)\;\Gamma(\upsilon+1)\;\Gamma(-\upsilon)\,\Gamma(\upsilon+\alpha-\gamma+\beta+1)}$$

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-\gamma} \; G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma-\upsilon-\alpha-\beta \\ 0, \alpha\end{matrix}\right] f(x)dx \qquad (2.9)$$

Provided that f(x) is such that (2.9) exists. Introducing (2.9) into (2.7), we obtain a general solution of (1.2) in the form

$$\theta(x,t) = 2^{\gamma-\alpha-\beta-1}\sum_{\upsilon=0}^{\infty} \frac{\Gamma(\alpha-\gamma+\beta+2\upsilon+1)\Gamma(\upsilon+\alpha+1)}{\Gamma(\beta+\upsilon-\gamma+1)\;\Gamma(\upsilon+1)\;\Gamma(-\upsilon)\,\Gamma(\upsilon+\alpha-\gamma+\beta+1)\Gamma(-\upsilon)}$$

$$\exp\{-k_\upsilon(\upsilon+\alpha-\gamma+\beta+1)\} \; G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma-\upsilon-\alpha-\beta \\ 0, -\alpha\end{matrix}\right]$$

$$\int_{-1}^{+1}(1-x)^{\alpha}\,(1+x)^{\beta-\gamma} \; G_{2,2}^{1,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon, \gamma-\upsilon-\alpha-\beta \\ 0, -\alpha\end{matrix}\right] f(x)\,dx,$$

(2.10)

Provided that the integral in (2.10) exists and the resulting series is convergent.

For f(x) given by (1.3) with the appeal of the result (1.7) we obtain the solution of the problem in the form.

$$\theta(x,t)= 2^{\rho+\sigma} \sum_{\upsilon=0}^{\infty} \sum_{N=0}^{\infty} \frac{(\alpha-\gamma+\beta+2\upsilon+1)_N(-\upsilon)_N\Gamma(\upsilon+1+\alpha)\Gamma(\alpha-\gamma+\beta+2\upsilon+1)}{N!\Gamma(\upsilon+1)\ \Gamma(\alpha+1+N)\Gamma(\upsilon-\gamma+\beta+1)\Gamma(-\upsilon)}$$

$$\exp\left\{-k_\upsilon(\upsilon+\alpha-\gamma+\beta+1)t\right\} G^{1,2}_{2,2}\left[\frac{1}{2}(x-1)\middle|\begin{matrix}1+\upsilon,\gamma-\upsilon-\alpha-\beta\\0,-\alpha\end{matrix}\right]$$

$$o,n_2:\ldots o,n_{r-1}:o,n_r+2:(m',n');\ldots..(m^{(r)},n^{(r)}$$

$$p_2,q_2:\ldots p_{r-1},q_{r-1}:p_{r+2},q_{r+1}\alpha:(p',q');\ldots;(p^{(r)},q^{(r)})$$

$$\left[\begin{matrix}(a_{2j};\alpha'_{2j};\alpha''_{2j})_{1,P_2}\ldots:(a_{\overline{r-1j}};\alpha'_{\overline{r-1j}},\ldots,\alpha^{r-1}_{\overline{r-1j}})1,p_{r-1}:\\ b_{2j},\beta'2j,\beta''_{2j})_{1,q_2}:\ldots:(b_{\overline{r-1j}};\beta'_{\overline{r-1j}},\ldots,\beta^{r-1}_{\overline{r-1j}})1,q_r-1:\end{matrix}\right.$$

$$(\gamma-\sigma-\beta;\mu_1,\ldots,\mu_r),\left(-\alpha-\rho-N;\delta_1,\ldots,\delta_r\right),$$

$$(b_{rj};\beta'_{rj},\ldots,\beta^r{}_{rj})_{1,q_r},(-\gamma-\alpha-\rho-\beta-\sigma-N-1;\delta_1+\mu_1\ldots\delta_r+\mu_r)$$

$$(a_{rj};\alpha'rj,\ldots,\alpha^{(r)}_{rj})_{1,p_r}:\left(a'_j,\alpha'_j\right)_{1,p'};\ldots;(a^{(r)}_j,\alpha^r_j)_{1,p^{(r)}}$$

$$\overline{}:\left(b'_j,\beta'_j\right)_{1,q'};\ldots;(b^{(r)}_j,\beta^r_j)_{1,q^r}\ |$$

$$\left.Z_1 2^{\delta_1+\mu_1},\ldots..,Z_r 2^{\delta r+\mu_r}\right]$$

provided

$$\operatorname{Re}(\rho+\alpha+\sum_{i=1}^{r}\delta_i\alpha_1)>-1,$$

$$\operatorname{Re}(\sigma+\beta+\sum_{i=1}^{r}\mu_1\alpha_1)>\gamma-1,|\arg Z_i|<\frac{1}{2}\pi U_i$$

$$(U_i>0),i=1\ldots..,r;\alpha_i,U_i\ (i=1,\ldots,r)$$

Being given by Prasad [2] and the series in (2.11) are convergent.

3. Particular Cases

(i) It is interesting as well as important to note that r =0 reduces the differential equation (1.2) into

$$\frac{1}{K}\frac{\partial\theta}{\partial t}=(1-x^2)\frac{\partial^2\theta}{\partial x^2}+[\beta-\alpha-(\alpha+\beta+2)x]\frac{\partial\theta}{\partial x}$$

(2.12)

And the equation (2.3) into the differential equation satisfied by Jacobi polynomial $P_{\upsilon}^{\alpha,\beta}(x)$.

(ii) When the bar moves with uniform velocity, the differential equation (1.2) reduces to that given Carslaw and Jeeger [3, p. 127(3)] with variable conductivity and no radiation.

(iii) when the bar is stationary, the differential equation (1.2) correspondence to that of given by Churchill [5, p. 224 (8)].

(iv) If we set $P_2=P_3 = \ldots = P_r = 0 = q_2 = q_3 = \ldots = q_{r-1}$ and $\gamma = O$, the result (2.11) reduces to the result recently obtained by Maurya (6).

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